Fractional Order Approach to Viscoelasticity in Dielectric Elastomers

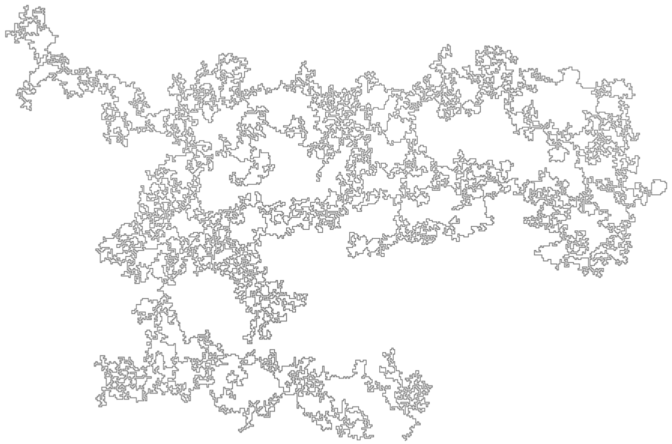
Paul Miles[1], Somayeh Mashayekhi[3], M. Yousuff Hussaini[3], William Oates[2]

[1]Department of Mathematics
North Carolina State University, Raleigh, NC 27695

[2]Department of Mechanical Engineering
[3]Department of Mathematics
Florida Center for Advanced Aero-Propulsion (FCAAP)
Florida A & M and Florida State University, Tallahassee, FL 32310

Fractal Structure-Fractional Properties

- Polymers often exhibit power-law constitutive relations
  - Mass: $M_D = M_0\left(\frac{R}{R_P}\right)^D$ $\quad D \leq 3$
  - Viscoelastic memory: $Q(t-\tau) = Q_0(t-\tau)^{\varepsilon-1}$
- Power law relations are compactly described by a fractional Taylor series



Fractal polymer network

Power Laws and Taylor Expansions

- Consider the power law function:

$$f(y) = q + (y - x)^p$$

- If p=2, a second order Taylor expansion will model this *exactly*
- What if p is a fraction?
  - The two term Caputo fractional Taylor series is *also exact*[1]
  - This has broad implications on development of continuum constitutive relations.
  - Will predictions beyond the calibrated regime improve?

1. Wheatcraft, S. W., Meerschaert, M. M. (2008). "Fractional conservation of mass." *Advances in Water Resources*, *31*(10), 1377-1381.

Dielectric Elastomers

- Applications:
  - Robotics
  - Flow control
  - Energy harvesting
  - Optical switches

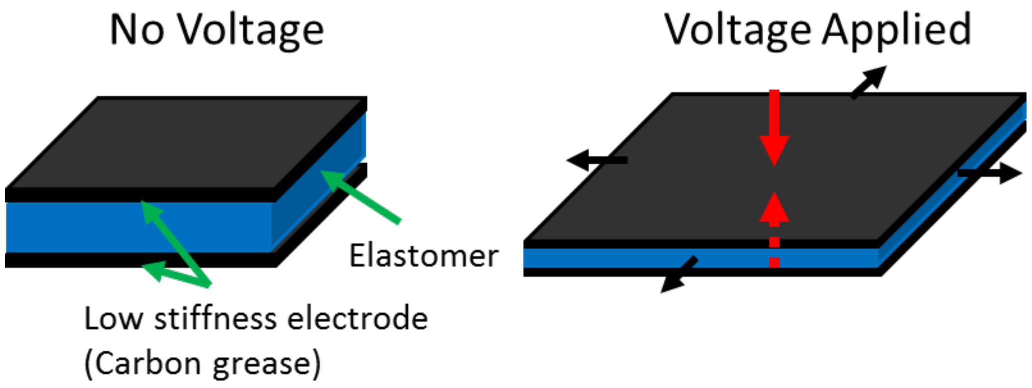


Figure: Out of plane expansion of elastomer as a result of transverse field.

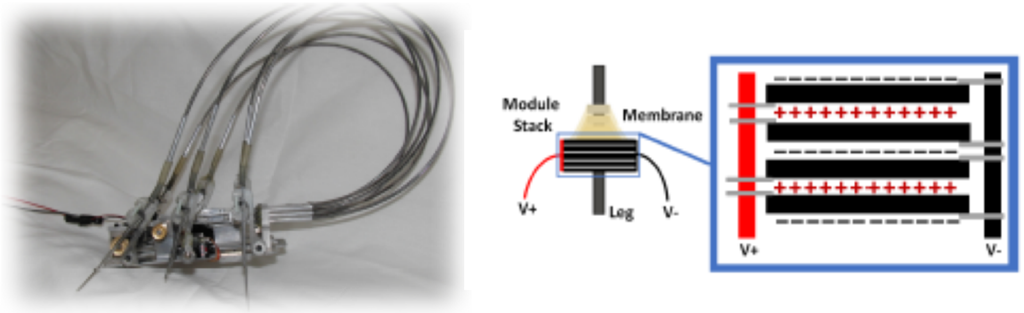


Figure: iSprawl robotic platform[1]. Membrane actuators control leg stiffness allowing for dynamic adaptation to different terrains.

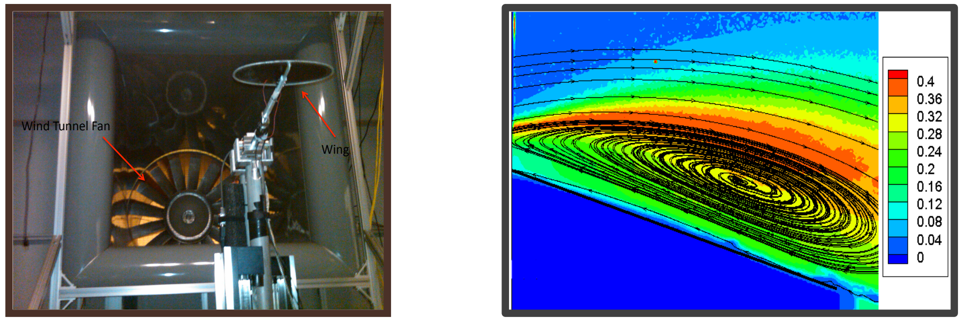


Figure: Wind tunnel experiments with deformable dielectric membrane[2]. By adjusting the membrane stiffness, the wing profile subsequently changed its shape and altered the flow characteristics.

1. Newton, J., “Design and Characterization of a Dielectric Elastomer Based Variable Stiffness Mechanism for Implementation onto a Dynamic Running Robot,” (2014), Figure 2.11 and Figure 4.5.
2. Hays, et al. "Aerodynamic Control of Micro Air Vehicle Wings Using Electroactive Membranes," J. Mater. Syst. Struct., v. 24(7), pp. 862-878, 2013.

Experimental Setup

- Very High Bond (VHB) 4910
- Specimens cycled
- Stretch rate: $\frac{\mathrm{d}\lambda}{dt} = \frac{1}{L_0}\frac{dx}{dt}$
  - $\frac{d\lambda}{dt}$ is the stretch rate (Hz)
  - $\frac{dx}{dt}$ is the speed of the moving clamp head (mm/s)
  - L_0 is the initial length of the VHB specimen (mm)

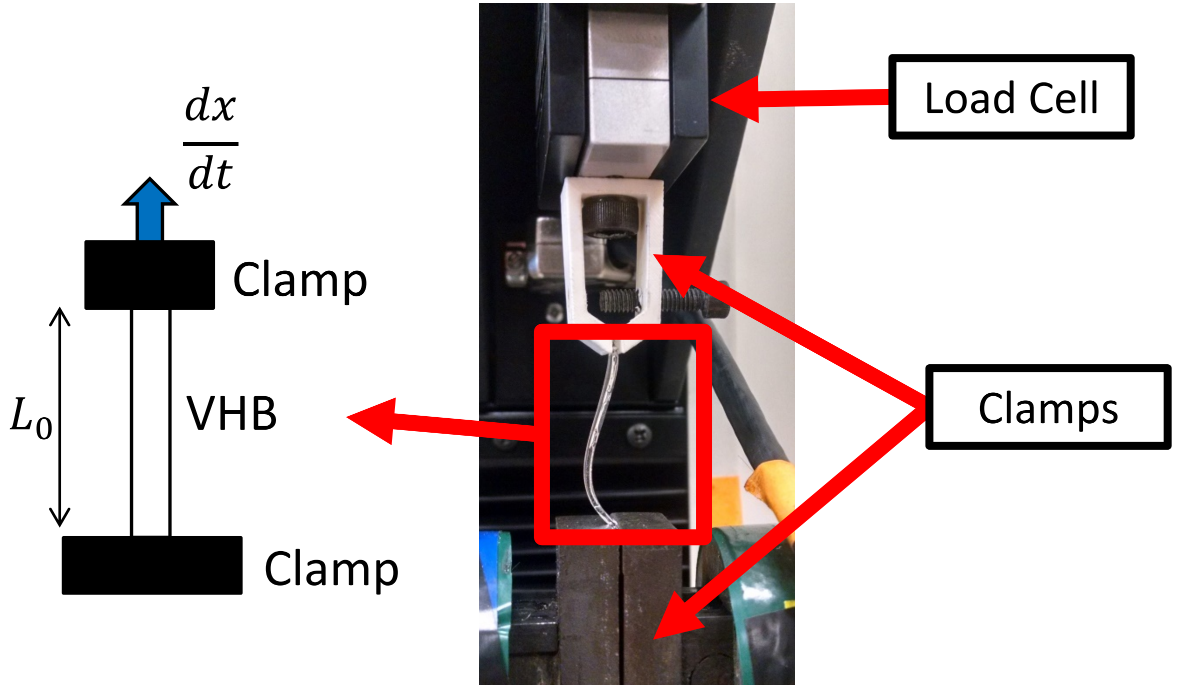
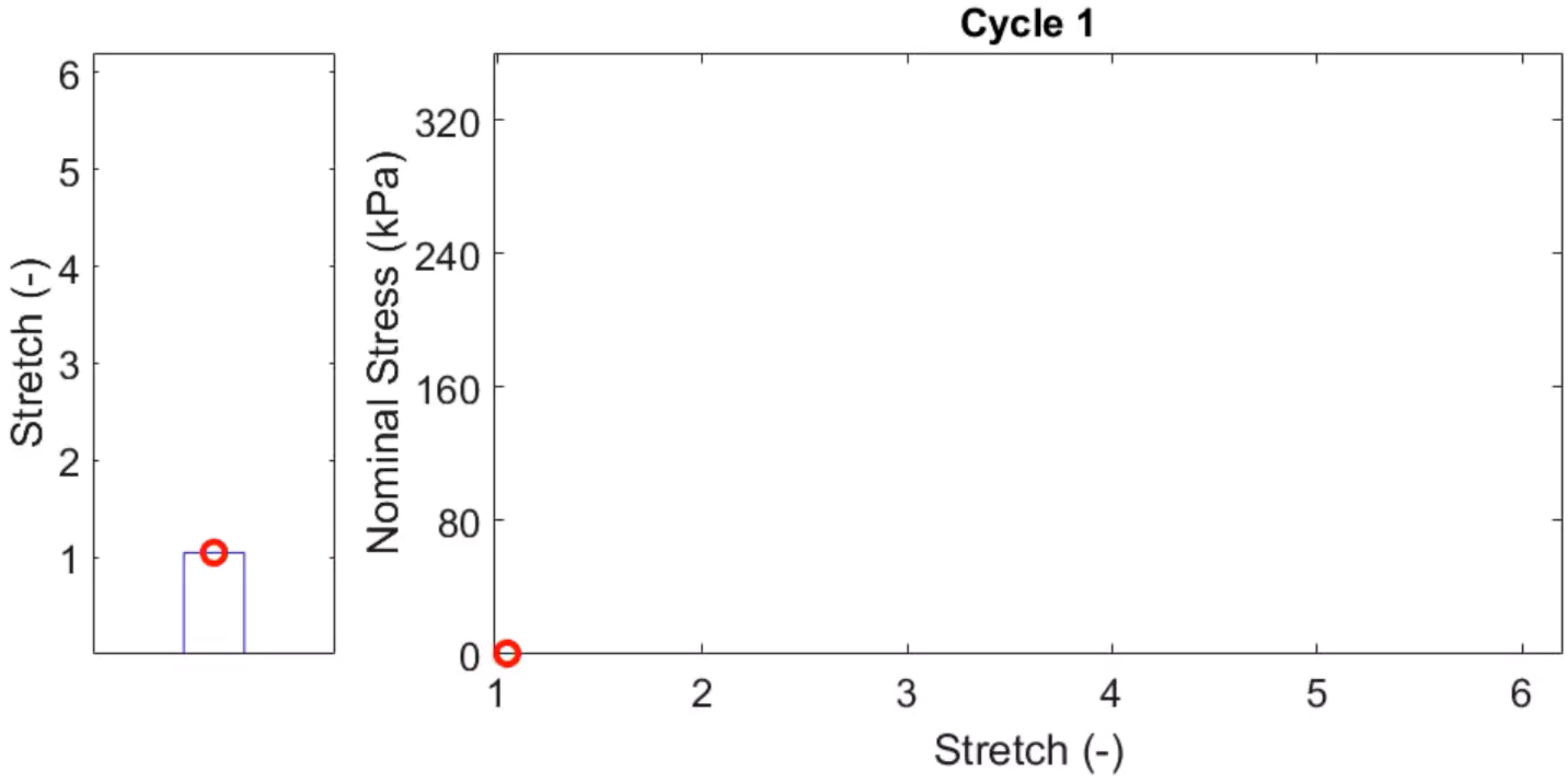


Figure: MTS tensile testing of VHB 4910

Experimental Observations



Experimental Observations

- Hysteresis
- Stress response decays
- Recovery
- Steady state

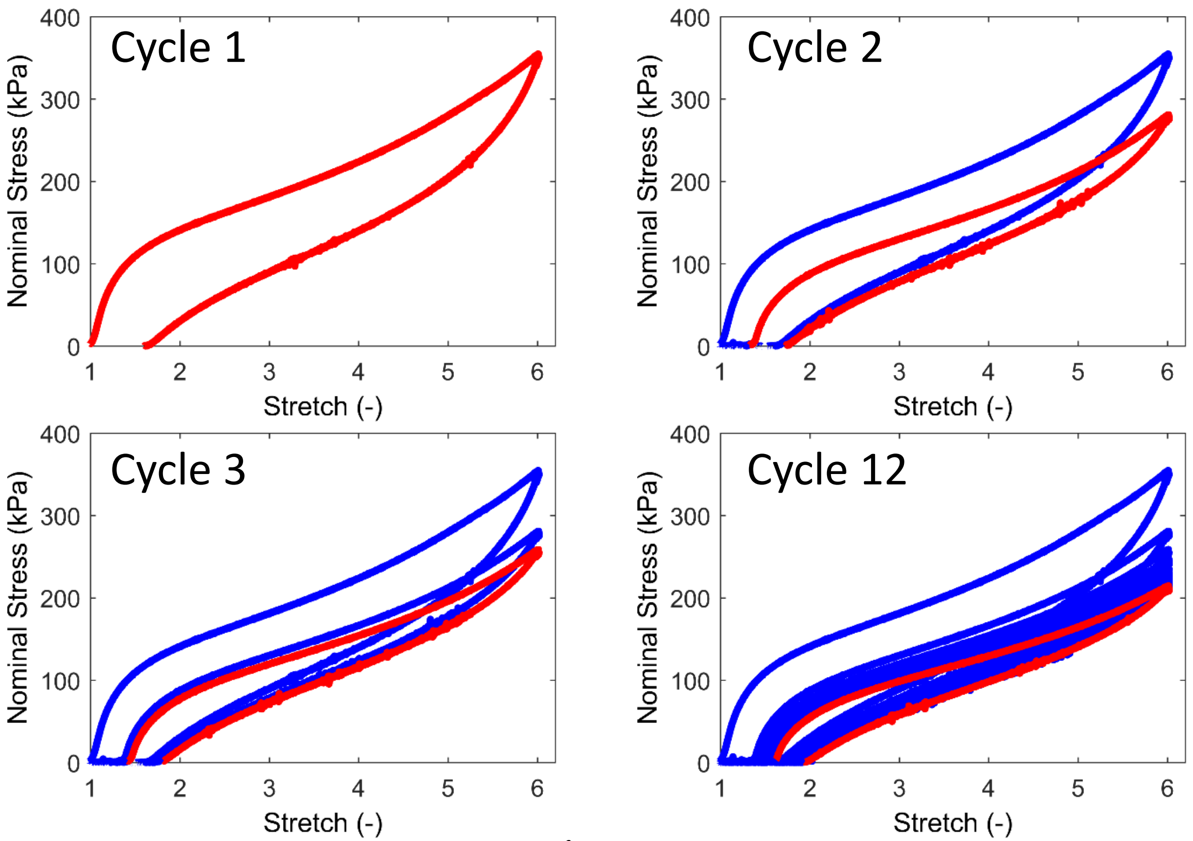


Figure: Cyclic loading of VHB 4910 at $\frac{d\lambda}{dt} = 0.67\ Hz$. Steady state hysteresis observed by the 12th cycle.

Experimental Observations

- Steady state hysteresis
- Rate-dependence
- Uncertainty
  - Measurement
  - Specimen variability

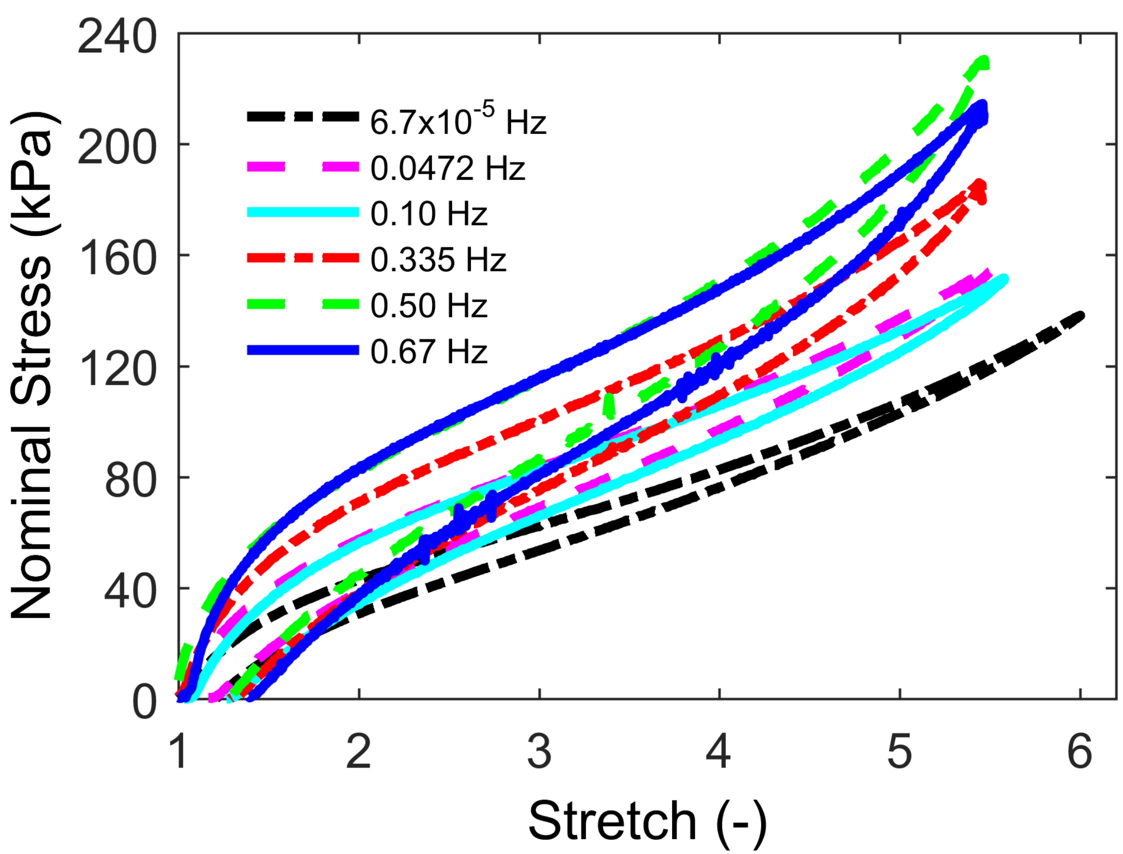


Figure: Steady state hysteresis loops from all stretch rates tested.

Theory: Viscoelasticity

- Total Energy Density

$$\psi = \psi_\infty(F_{iK}, \Theta) + \Upsilon(F_{iK}, \Theta, \Gamma^\nu_{iK})$$

- Nominal Stress

$$s_{iK} = \frac{\partial\hat{\psi}}{\partial F_{iK}} = \frac{\partial\psi_\infty}{\partial F_{iK}} - pJH_{iK} + \frac{\partial\Upsilon}{\partial F_{iK}}$$

- Viscoelastic Stress

$$Q^\nu_{iK} = -\frac{\partial\hat{\psi}}{\partial\Gamma^\nu_{iK}} = -\frac{\partial\Upsilon}{\partial\Gamma^\nu_{iK}}$$

- Components
  - ψ_∞ - conserved, hyperelastic energy function
  - Υ - energy function that depends on non-conserved internal state
  - F_{iK} - deformation gradient
  - Θ - temperature
  - Γ^ν_{iK} - set of non-conserved internal strains

Theory: Viscoelasticity

- Conserved, hyperelastic energy function[1]

$$\psi_\infty = \frac{1}{6} G_c I_1 - G_c \lambda_{max}^2 \log(3\lambda_{max}^2 - I_1) + G_e \sum_j (\lambda_j + \frac{1}{\lambda_j})$$

- Non-conserved energy functions: Linear and Non-linear[2,3]

$$\Upsilon_L = \sum_\nu [\frac{1}{2} \gamma_\nu (F_{iK} - \Gamma_{iK}^\nu)(F_{iK} - \Gamma_{iK}^\nu)]$$

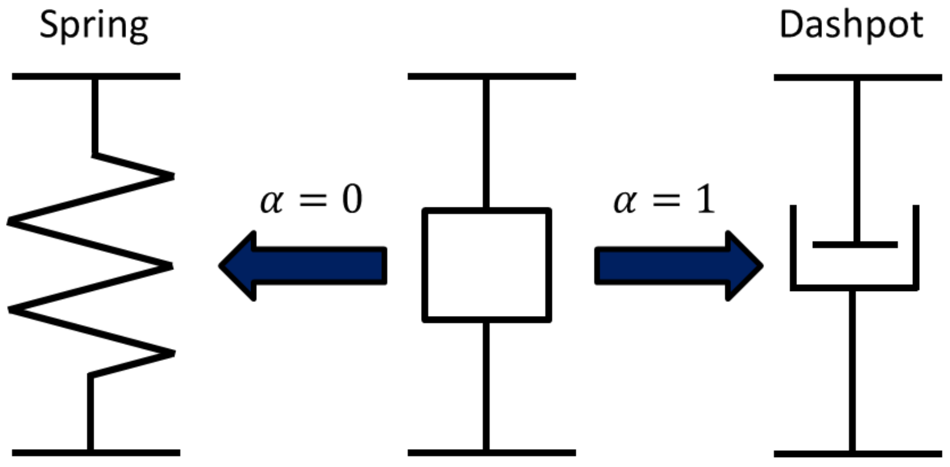
$$\Upsilon_{NL} = \sum_\nu [\frac{1}{2} \gamma_\nu \Gamma_{iK}^\nu \Gamma_{iK}^\nu - \beta_\infty^\nu \frac{\partial \psi_\infty}{\partial F_{iK}} \Gamma_{iK}^\nu + \beta_\infty^\nu \psi_\infty]$$

- Components
  - G_c - Crosslink modulus
  - G_e - Entanglement modulus
  - λ_{max} - Maximum extension of affine tube
  - β_∞^ν - Phenomenological set of parameters
  - γ_ν - Proportional to viscosity of polymer network

1. Davidson, J. D., Goulbourne, N.C. "A nonaffine network model for elastomers undergoing finite deformations." Journal of the Mechanics and Physics of Solids 61.8 (2013): 1784-1797.
2. Holzapfel & Simo, Int. J. Solid Struct., (1996), v. 33(20-22), pp. 3019-3034.
3. Miles, P., Hays, M., Smith, R., Oates, W. “Bayesian Uncertainty Analysis of Finite Deformation Viscoelasticity.” *Mechanics of Materials*, 2015, Vol. 91, pp. 35-49.

Theory: Viscoelasticity

- Integer order
  - Model limitation
  - Parameters are rate dependent[1]
- Fractional order
  - Replace standard calculus operators with fractional order operators



$Q_{iK} = \gamma F_{iK}$ $\quad$ $Q_{iK} = \eta D_t^{\alpha} F_{iK}$ $\quad$ $Q_{iK} = \eta \dfrac{dF_{iK}}{dt}$

Figure: Fractional order spring-dashpot system. As the fractional order approaches 0 the system behaves like a dashpot. Likewise, as the fractional order approaches 1 the system behaves like a spring.

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(s)}{(t-s)^{\alpha+1-n}} ds, \qquad n-1 < \alpha \leq n, \qquad n \in \mathbb{N}$$

1. Miles, P., Hays, M., Smith, R., Oates, W. "Bayesian Uncertainty Analysis of Finite Deformation Viscoelasticity." *Mechanics of Materials*, 2015, Vol. 91, pp. 35-49.

Uncertainty Quantification: Bayesian Statistical Analysis

- Statistical Model: $M^{data}(i) = M(i;\theta) + \varepsilon_i, \quad i = 1, \ldots, N$
- Bayes' Relation

$$\pi(\theta | M^{data}) = \frac{p(M|\theta)\pi_0(\theta)}{\int_{\mathbb{R}_p} p(M|\theta)\,\pi_0(\theta)d\theta}$$

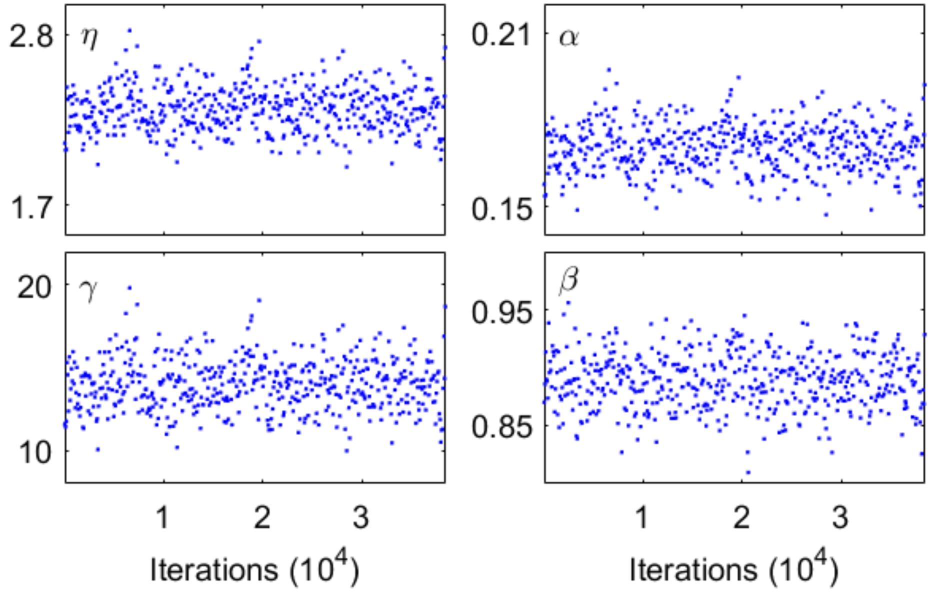
- Posterior Density: $\pi(\theta | M^{data})$
- Prior Density: $\pi_0(\theta)$
- Likelihood Function: $p(M|\theta) = \ e^{-\sum_{i=1}^{n}[M^{data}(i) - M(i;\theta)]^2/(2\sigma^2)}$
  - Assume observation errors are independent and identically distributed (iid): $\varepsilon_i \sim N(0, \sigma^2)$.

Parameter Estimation:

- Calibrated with data:

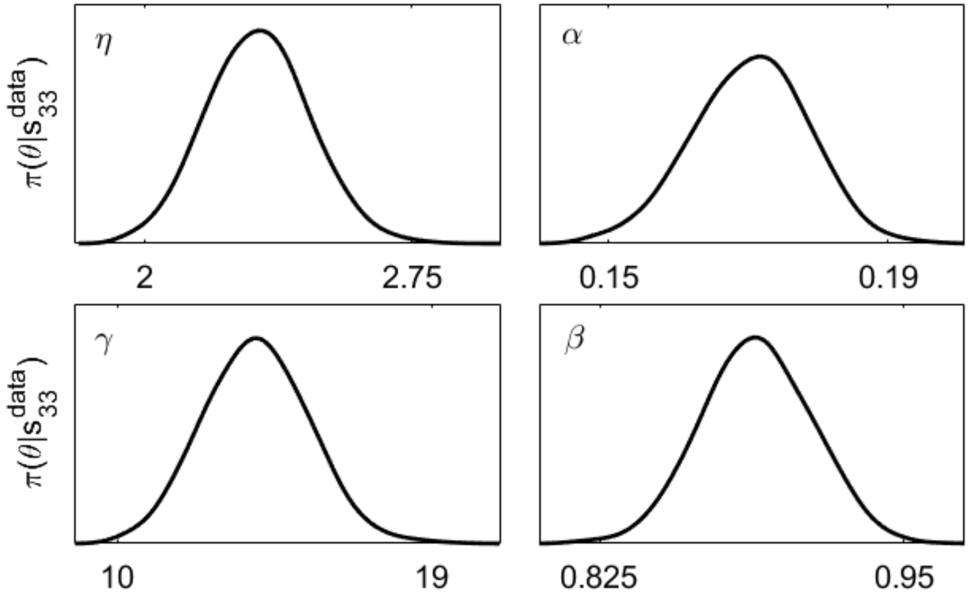
$$\frac{d\lambda}{dt} = 0.67\ Hz$$

- Parameter chains



- Posterior densities

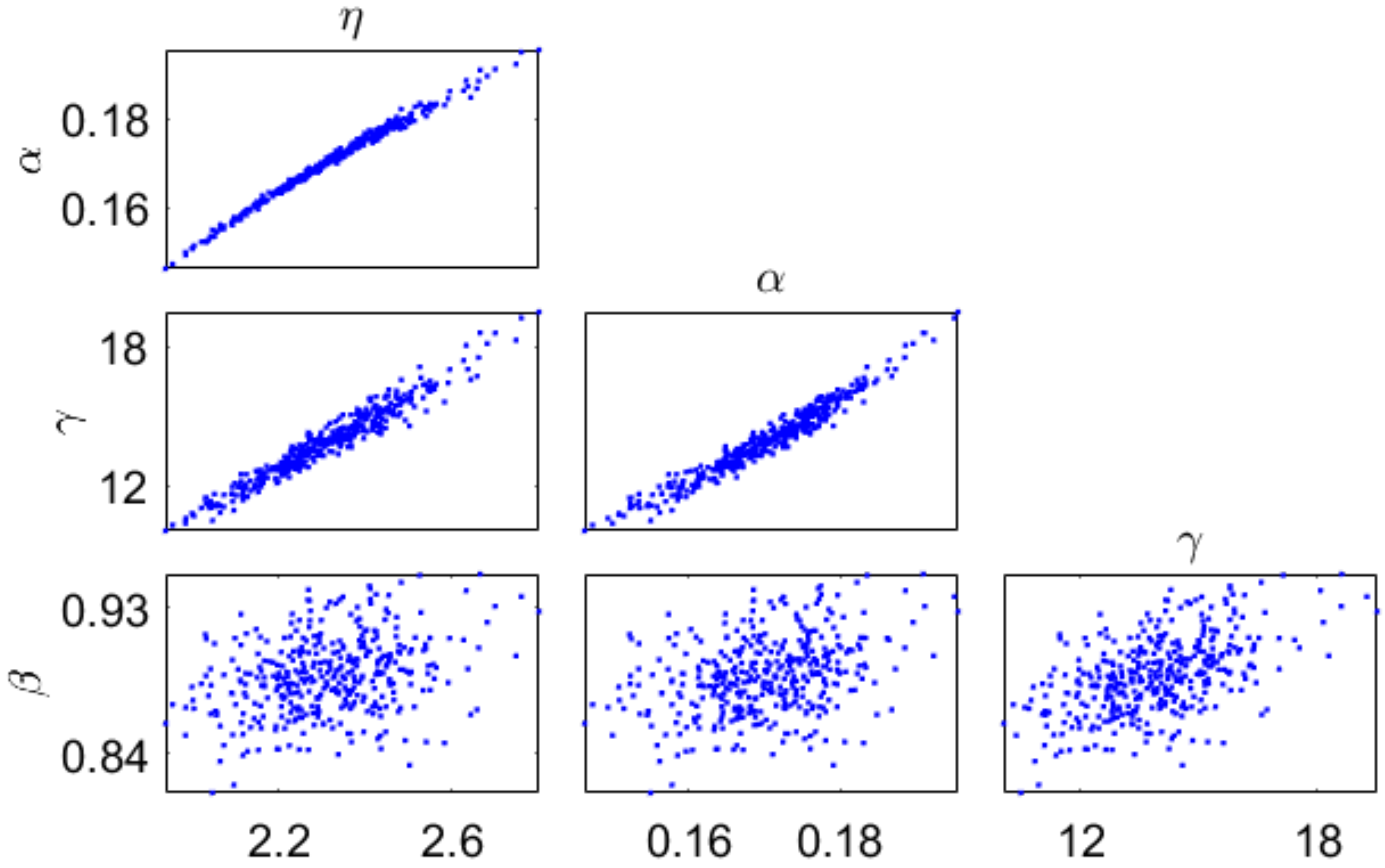
$$\pi(\theta|M^{data})$$



- Convergence:
  - “Burned-In”

Parameter Correlation: Viscoelasticity

- Pairwise correlation
- Nearly single valued
- Sensitivity



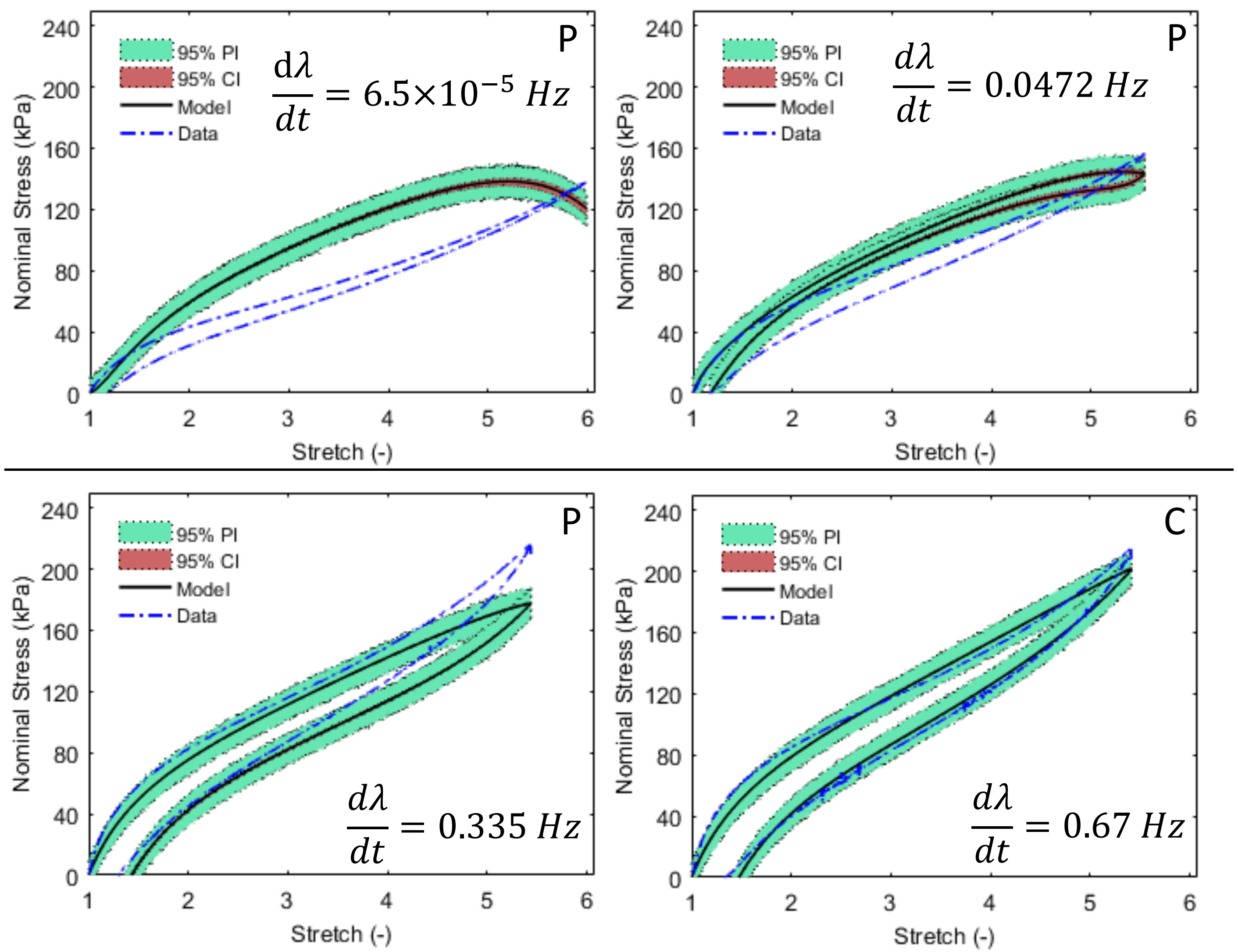
Uncertainty Propagation: Viscoelasticity

- C – Calibrated
- P – Predicted

- Non-linear viscoelasticity
- Integer order rate relation

$$\frac{dQ_{iK}}{dt} + \frac{\gamma}{\eta} Q_{iK} = \beta \frac{ds_{iK}^{\infty}}{dt}$$

Calibrated Rate (1/s)	$\bar{\eta}$ (kPa·s)	$\bar{\gamma}$ (kPa)	$\bar{\beta}$ (-)
0.67	5.2×10^2	118	6.42



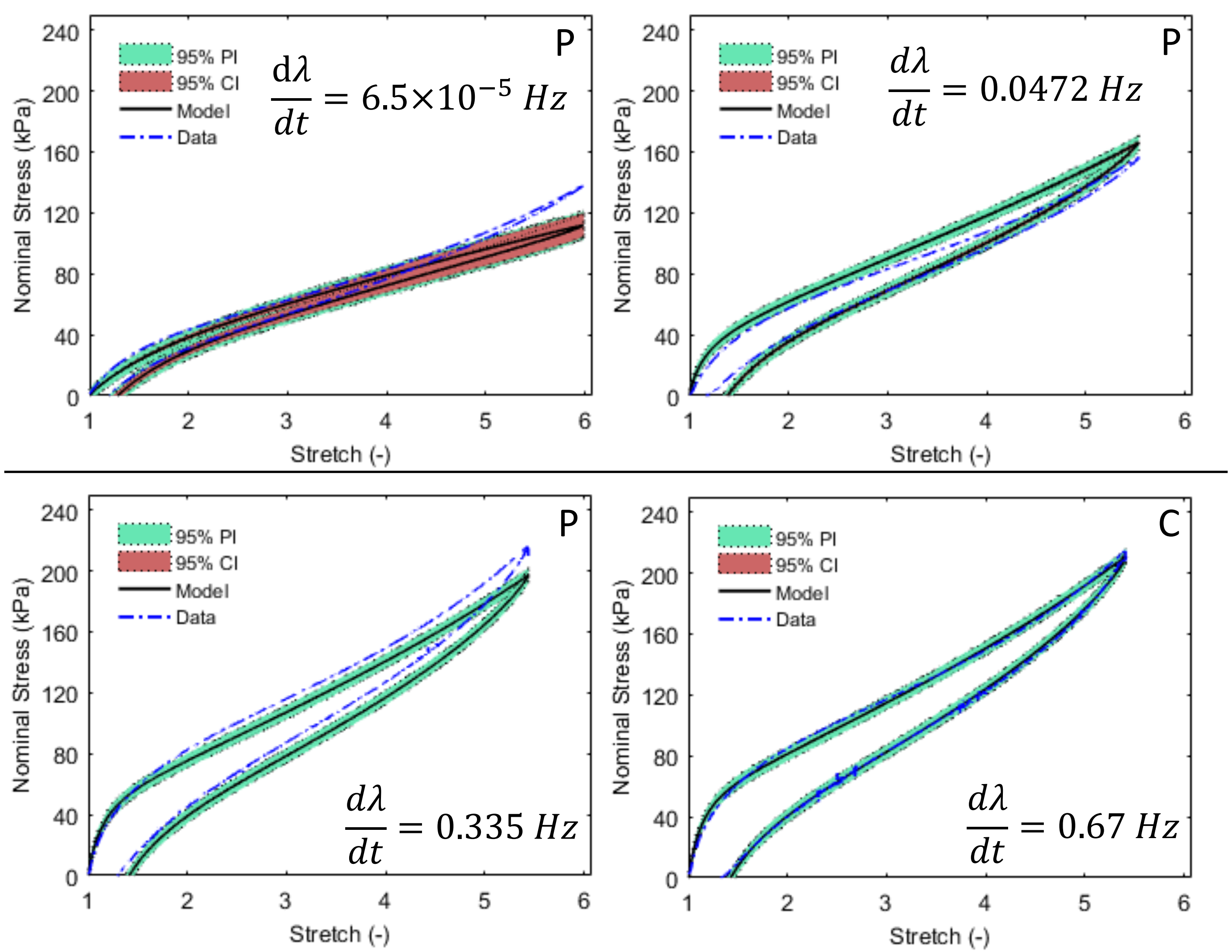
Uncertainty Propagation: Viscoelasticity

- C – Calibrated
- P – Predicted

- Non-linear viscoelasticity
- Fractional order rate relation

$$Q_{iK} = {\color{red}\eta} D_t^{\color{red}\alpha} s_{iK}^{\infty}$$

Calibrated Rate (1/s)	$\bar{\eta}$ (kPa·s)	$\bar{\gamma}$ (kPa)	$\bar{\beta}$ (-)	$\bar{\alpha}$ (-)
0.67	2.32	14.0	0.89	0.17



Conclusions

- Fractional order viscoelasticity applied to VHB elastomers
  - Predicts behavior over broad range of operating regimes
  - Parameters are independent of rate
- Bayesian statistical analysis established a fractional order of ~0.17 for VHB 4910
  - This fractional order is expected to be material dependent

Acknowledgements

Support:

- P.M. gratefully acknowledges support from an Army Research Collaborative Technology Alliance (R-CTA) and the Army Research Office through grant W911NF-13-1-0146, program manager Matt Munson. This work has also been supported by NSF CDS&E awards (CMMI grants 1306320 and 1306290).